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MAX PLANCK INSTITUTE
FOR SECURITY AND PRIVACY

Introduction to lattice-based KEMs

May 4, 2022

Count of Problem Category	Column Labels		
Row Labels	Key Exchange	Signature	Grand Total
?	1		1
Braids	1	1	2
Chebychev	1		1
Codes	19	5	24
Finite Automata	1	1	2
Hash		4	4
Hypercomplex Numbers	1		1
Isogeny	1		1
Lattice	24	4	28
Mult. Var	6	7	13
Rand. walk	1		1
RSA	1	1	2
Grand Total	**57**	**23**	**80**

Overview tweeted by Jacob Alperin-Sheriff on Dec 4, 2017.

All the way back in 2016. . .

Google Security Blog

The latest news and insights from Google on security and safety on the Internet

Experimenting with Post-Quantum Cryptography

July 7, 2016

Posted by Matt Braithwaite, Software Engineer

Search blog ...

Archive

"We're indebted to Erdem Alkim, Léo Ducas, Thomas Pöppelmann and Peter Schwabe, the researchers who developed "New Hope", the post-quantum algorithm that we selected for this experiment."

https://security.googleblog.com/2016/07/experimenting-with-post-quantum.html

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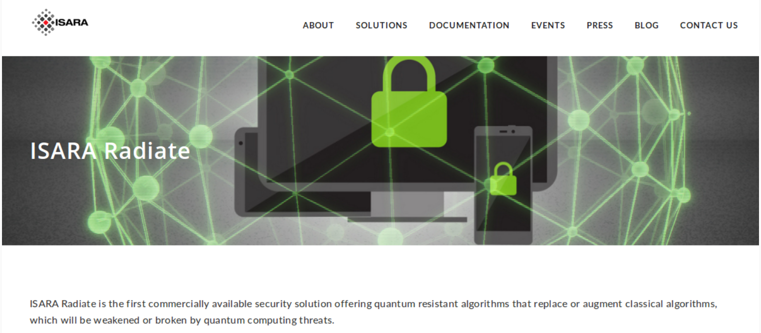

“Key Agreement using the ‘NewHope’ lattice-based algorithm detailed in the New Hope paper, and LUKE (Lattice-based Unique Key Exchange), an ISARA speed-optimized version of the NewHope algorithm.”

https://www.isara.com/isara-radiate/

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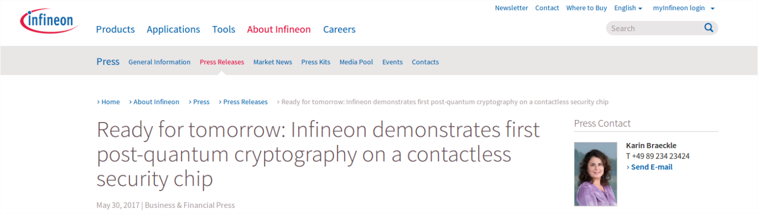

"The deployed algorithm is a variant of "New Hope", a quantum-resistant cryptosystem"

https://www.infineon.com/cms/en/about-infineon/press/press-releases/2017/INFCCS201705-056.html

Learning with errors (LWE)

- Given uniform $\mathbf{A} \in \mathbb{Z}_q^{k \times \ell}$
- Given “noise distribution” χ
- Given samples $\mathbf{A}\mathbf{s} + \mathbf{e}$, with $\mathbf{e} \leftarrow \chi$

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  - Kyber/Saber: use small-dimension matrices and vectors over $\mathcal{R}_q = \mathbb{Z}_q[X]/(X^{256} + 1)$
- Perform arithmetic on (vectors of) polynomials instead of vectors/matrices over $\mathbb{Z}_q$

How to build a KEM?

Alice (server)		Bob (client)
$\mathbf{s}, \mathbf{e} \xleftarrow{\$} \chi$		$\mathbf{s}', \mathbf{e}' \xleftarrow{\$} \chi$
$\mathbf{b} \leftarrow \mathbf{a}\mathbf{s} + \mathbf{e}$	$\xrightarrow{\mathbf{b}}$	$\mathbf{u} \leftarrow \mathbf{a}\mathbf{s}' + \mathbf{e}'$
	$\xleftarrow{\mathbf{u}}$	

$$\begin{aligned} \text{Alice has} \quad & \mathbf{v} && = \mathbf{u}\mathbf{s} && = \mathbf{a}\mathbf{s}\mathbf{s}' + \mathbf{e}'\mathbf{s} \\ \text{Bob has} \quad & \mathbf{v}' && = \mathbf{b}\mathbf{s}' && = \mathbf{a}\mathbf{s}\mathbf{s}' + \mathbf{e}\mathbf{s}' \end{aligned}$$

- Secret and noise polynomials $\mathbf{s}, \mathbf{s}', \mathbf{e}, \mathbf{e}'$ are small
- $\mathbf{v}$ and $\mathbf{v}'$ are *approximately* the same

How to build a KEM, part 2

Alice		Bob
$\mathbf{s}, \mathbf{e} \xleftarrow{\$} \chi$		$\mathbf{s}', \mathbf{e}' \xleftarrow{\$} \chi$
$\mathbf{b} \leftarrow \mathbf{a}\mathbf{s} + \mathbf{e}$	$\xrightarrow{(\mathbf{b}\quad)}$	
		$\mathbf{u} \leftarrow \mathbf{a}\mathbf{s}' + \mathbf{e}'$
		$\mathbf{v} \leftarrow \mathbf{b}\mathbf{s}'$
$\mathbf{v}' \leftarrow \mathbf{u}\mathbf{s}$	$\xleftarrow{(\mathbf{u}\quad)}$	

How to build a KEM, part 2

Alice		Bob
$\mathit{seed} \xleftarrow{\$} \{0,1\}^{256}$		
$\mathbf{a} \leftarrow \mathrm{Parse}(\mathrm{XOF}(\mathit{seed}))$		
$\mathbf{s}, \mathbf{e} \xleftarrow{\$} \chi$		$\mathbf{s}', \mathbf{e}' \xleftarrow{\$} \chi$
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		$\mathbf{u} \leftarrow \mathbf{as}' + \mathbf{e}'$
		$\mathbf{v} \leftarrow \mathbf{bs}'$
		$k \xleftarrow{\$} \{0,1\}^n$
		$\mathbf{k} \leftarrow \mathsf{Encode}(k)$
$\mathbf{v}' \leftarrow \mathbf{us}$	$\xleftarrow{(\mathbf{u}, \mathbf{c})}$	$\mathbf{c} \leftarrow \mathbf{v} + \mathbf{k}$

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$\mathbf{k}' \leftarrow \mathbf{c} - \mathbf{v}'$		$\mu \leftarrow \mathrm{Extract}(\mathbf{k})$
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$\mathbf{k}' \leftarrow \mathbf{c} - \mathbf{v}'$		$\mu \leftarrow \mathsf{Extract}(\mathbf{k})$
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This is LPR encryption, written as KEM (except for generation of $\mathbf{a}$)

Encode and Extract

- Encoding in LPR encryption: map n bits to n coefficients:
  - A zero bit maps to 0
  - A one bit maps to $q/2$
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- Decode: map coefficient into $[-q/2, q/2]$
    - Closer to 0 (i.e., in $[-q/4, q/4]$): set bit to zero
    - Closer to $\pm q/2$: set bit to one

From passive to CCA security

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- The base scheme does not have active security
- Attacker can choose arbitrary noise, learns $\mathbf{s}$ from failures
- Fujisaki-Okamoto transform (sketched):

Alice (Server)		Bob (Client)
<u>Gen():</u> pk, sk$\leftarrow$KeyGen()		<u>Enc(seed, $\mathbf{b}$):</u>
seed, $\mathbf{b}$$\leftarrow$pk	$\xrightarrow{\text{seed}, \mathbf{b}}$	$x \leftarrow \{0, \ldots, 255\}^{32}$ k, coins$\leftarrow$SHA3-512(x)
	$\xleftarrow{\mathbf{u}, v}$	$\mathbf{u}, v$$\leftarrow$Encrypt((seed, $\mathbf{b}$), x, coins)
<u>Dec($\mathbf{s}$, ($\mathbf{u}$, v)):</u> $x' \leftarrow$ Decrypt($\mathbf{s}$, ($\mathbf{u}$, v)) k', $coins'$$\leftarrow$SHA3-512($x'$) $\mathbf{u}'$, $v'$$\leftarrow$Encrypt((seed, $\mathbf{b}$), x', coins$'$) verify if $(\mathbf{u}', v') = (\mathbf{u}, v)$		

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- Encrypt:
  - Map message m to $\mathbf{m} \in \mathcal{R}_q$ with coefficients in $\{-1, 0, 1\}$
  - Sample random small-coefficient polynomial $\mathbf{r} \in \mathcal{R}_q$
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  - Compute $\mathbf{m} = \mathbf{v} \cdot \mathbf{f}_p \mod p$
- Advantages/Disadvantages compared to LPR:
  - Asymptotically weaker than Ring-LWE approach
  - Slower keygen, but faster encryption/decryption

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- "Traditionally", work directly with elements of $\mathcal{R}_q$ ("Ring-LWE")
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  - Choose smaller *n*, e.g., $n = 256$ (Kyber, Saber, ThreeBears)
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- MLWE can very easily scale security (change dimension of matrix):
  - Optimize arithmetic in $\mathcal{R}_q$ once
  - Use same optimized $\mathcal{R}_q$ arithmetic for all security levels

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  - Fixed-weight noise or not?
    - Fixed-weight noise needs random permutation (sorting)
    - Naive implementations leak secrets through timing
    - Advantage of fixed-weight: easier to bound (or eliminate) decryption failures

Design space 4: allow failures?

- Can avoid decryption failures entirely (NTRU, NTRU Prime)
- Advantage:
  - Easier CCA security transform and analysis
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- Solution in NewHope: Choose a fresh $\mathbf{a}$ every time
- Server can cache $\mathbf{a}$ for some time (e.g., 1h)
- All NIST PQC candidates now use this approach

Design space 6: error-correcting codes?

- Ring-LWE/LWR schemes work with polynomials of > 256 coefficients
- "Encrypt" messages of > 256 bits
- **Need to encrypt** only 256-bit key
- Question: How do we put those additional bits to use?
- Answer: Use error-correcting code (ECC) to reduce failure probability

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- Answer: Use error-correcting code (ECC) to reduce failure probability
- NewHope: very simple threshold decoding
- LAC, Round5: more advanced ECC
    - Correct more errors, obtain smaller public key and ciphertext
    - More complex to implement, in particular without leaking through timing

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  - More flexibility for secret/noise generation
- Disadvantages:
  - Less robust (will somebody reuse keys?)
  - More options (CCA vs. CPA): easier to make mistakes

Design space 8: CCA transforms

- General Fujisaki-Okamoto principle is the same for most KEMs (exception: NTRU)
- Tweaks to FO transform:
    - Hash public-key into coins: multitarget protection (for non-zero failure probability)

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- As of round 2, no proposal uses explicit rejection
  - Would break some security reduction
  - More robust in practice (return value alwas `0`)

Summary

- Lattice-based KEMs offer best overall performance in the PQ world
- Many tradeoffs between
  - Security (including passive vs. active)
  - Failure rate
  - Size
  - Speed
- More information about NIST PQC:
  - `https://csrc.nist.gov/projects/post-quantum-cryptography`
  - `https://pqc-wiki.fau.edu/`

Exercise: the Wookie encapsulation mechanism

Download `https://cryptojedi.org/wookie.tar.gz`
Slides at `https://cryptojedi.org/latticekems.pdf`

- CPA-secure "LPR KEM", see slide 7
- Work in polynomial ring $\mathcal{R}_q = \mathbb{Z}_q[X]/(X^n + 1)$
- Parameters $q = 4096$, $n = 1024$
- Centered binomial noise with $k = 8$
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1. Implement arithmetic in $\mathcal{R}_q$ (file `poly.c`)
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- `make` builds various unit tests in `test/` subdirectory
- Running `test.sh` in `test/` subdirectory runs all tests

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- Resulting coefficient will be in $\{-8, ..., 8\}$
- Sampling a polynomial needs $2n = 2048$ uniformly random bytes

Some remarks

- Software skeleton assumes Linux system
- Need basic build tools (`make`, `gcc`, . . .) installed:

```
apt install build-essential
```

- Some unit tests and `test.sh` script assume Sage to be installed

```
apt install sagemath
```

- Can also download pre-compiled binaries of Sage:
  https://doc.sagemath.org/html/en/installation/binary.html