How to use the negation map in the Pollard rho method

Peter Schwabe
joint work with Daniel J. Bernstein and Tanja Lange

National Taiwan University

June 16, 2011

Crypto Séminaire
Université de Versailles Saint-Quentin-en-Yvelines

A few words about Taiwan and NTU

- Taiwan (台灣) is an island south of China
- About 36,200 km^2 large
- Territory of the Republic of China (not to be confused with the People's Republic of China)
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- If you are curious: We host PQCrypto in November this year (submission deadline is June 24)

A picture from Taiwan – Sun-Moon Lake (日月潭)

For more pictures check out http://cryptojedi.org/gallery/

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$$k \cdot P = Q$$
- For certain groups G this problem is the basis of many asymmetric cryptographic protocols
- Most importantly: $\mathbb{Z}/n\mathbb{Z}$ and elliptic-curve groups

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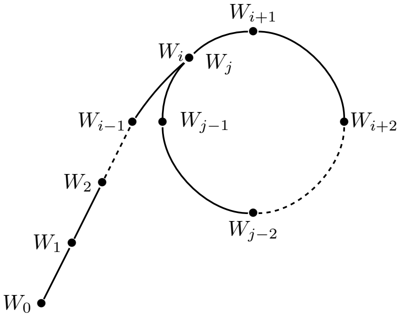
- f needs to preserve knowledge about the linear combination in P and Q
- If $W_i = W_j$ for $i \neq j$, then

$$\begin{aligned} & n_i P + m_i Q = n_j P + m_j Q \Rightarrow \\ & k = (n_j - n_i)/(m_i - m_j) \mod |G| \end{aligned}$$

Pollard's rho algorithm II



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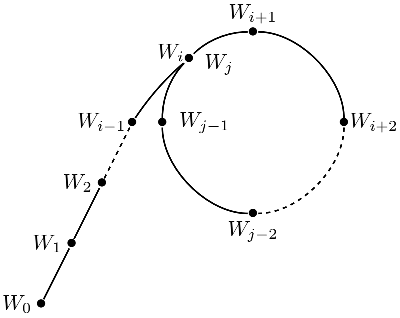

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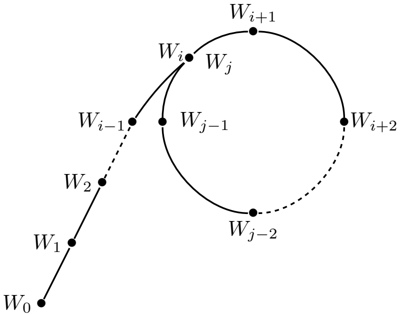

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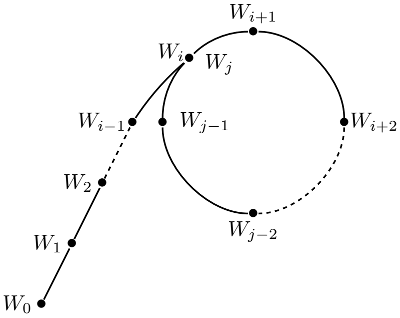

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- Detect cycles without storing all W_i: Floyd, Brent

Parallel Pollard

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- Server searches in incoming points for collisions (same DP, different starting point)

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- Choice of DP-property influences length of separate walks
- Fewer DPs: longer walks (on average), less storage, less communication
- More DPs: Less overhead after a collision
- Clients do not have to update n_i and m_i, simply do successful walks again to find coefficients

Additive walks

- Main cost of (parallalized) Pollard's rho algorithm: calls to the iteration function
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- Precompute r pseudorandom elements $R_0, \dots, R_{r-1}$ with known linear combination in P and Q
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- Summary: additive walks provide much better performance in practice

Application to elliptic-curve groups

- So far, everything worked in the generic-group model
- Now consider groups of points on elliptic curves
- Group elements are points (x, y)
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- Idea: Define iterations on equivalence classes modulo negation
- For example: always take the lexicographic minimum of $(x, -y)$ and (x, y)
- This halves the size of the search space, expected number of iterations drops by a factor of $\sqrt{2}$

Putting it together

- Precompute $R_0, \ldots, R_{r-1}$
- Clients start at some random W_0
- Iteratively compute $W_{i+1} = |W_i + R_{h(W_i)}|$
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- Probability for such fruitless cycles: $1/2r$
- Similar observations hold for longer fruitless cycles (length 4,6,...)
- Probability of a cycle of length $2c$ is $\approx 1/r^c$

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Escape strategies

- Retroactively adjust $h(W_i)$
- Determine unique point in cycle, add "special point" to escape
- Determine unique point in cycle, double this point
- Important: Escape point must be independent from entrance point

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  > *"If the Pollard rho method is parallelized in SIMD fashion, it is a challenge to achieve any speedup at all. . . . Dealing with cycles entails administrative overhead and branching, which cause a non-negligible slowdown when running multiple walks in SIMD-parallel fashion. . . . [This] is a major obstacle to the negation map in SIMD environments."*

What's the problem with SIMD?

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- Question: Can we really not get the factor-$\sqrt{2}$ speedup with SIMD?

Our approach

- Solve ECDLP on elliptic curve over $\mathbb{F}_p$
- Begin with simplest type of negating additive walk
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  - Otherwise set $W_i = W_{i-1}$
- With even lower frequency check for 4-cycles, 6-cycles etc.
- Implementation actually checks for 12-cycles (with very low frequency)

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- All selections, subtractions, additions and comparisons are linear-time
- Asymptotalically negligible compared to finite-field multiplications in EC arithmetic

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Optimization and analysis

- Checking for fruitless cycles every w iterations
- Probability for fruitless cycle: $w/2r$
- Average wasted iterations if fruitless cycle occured: $w/2$
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- Negligible if $r \to \infty$ as $p \to \infty$

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- Paper is online, e.g. at http://cryptojedi.org/papers/#negation