Pairings I

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ECC Summer School 2008, Eindhoven
18 September 2008

What is a pairing?

A pairing is a non-degenerate, bilinear map

$$e : G_1 \times G_2 \to G_3,$$

where G_1, G_2 are abelian groups written additively and G_3 is a multiplicative abelian group.

- Non-degenerate:
  for all $0 \neq P \in G_1$ there is a $Q \in G_2$ s.t. $e(P, Q) \neq 1$,
  for all $0 \neq Q \in G_2$ there is a $P \in G_1$ s.t. $e(P, Q) \neq 1$.
- Bilinear: for $P_1, P_2 \in G_1; Q_1, Q_2 \in G_2$ we have

$$\begin{aligned} e(P_1 + P_2, Q_1) &= e(P_1, Q_1)e(P_2, Q_1), \\ e(P_1, Q_1 + Q_2) &= e(P_1, Q_1)e(P_1, Q_2). \end{aligned}$$

It follows: $e([a]P, [b]Q) = e(P, Q)^{ab} = e([b]P, [a]Q)$.

What can be done with pairings?

Pairings on elliptic curves can be used,

- as a means to attack DL-based cryptography on groups of points on elliptic curves,
- or to construct crypto systems with certain special properties:
  - One-round tripartite key agreement,
  - Identity-based key agreement,
  - Identity-based encryption (IBE),
  - Hierarchical IBE (HIDE),
  - Short signatures (BLS).
  - much more ...

Elliptic curves

Let $p > 3$ be a prime, $\mathbb{F}_p$ the finite field with p elements and

$$E : Y^2 = X^3 + AX + B$$

an elliptic curve over $\mathbb{F}_p$.

- For a field extension $\overline{\mathbb{F}_p} \supseteq L \supseteq \mathbb{F}_p$ let

  $$E(L) = \{(x, y) \in L^2 : y^2 = x^3 + Ax + B\} \cup \{P_\infty\}$$

  the group of L-rational points on E.
- Let $n = \#E(\mathbb{F}_p)$ be the number of $\mathbb{F}_p$-rational points. We have

  $$n = p + 1 - t, \quad |t| \leq 2\sqrt{p},$$

  where t is the trace of Frobenius.

Torsion points

Let m be a non-negative integer. The set of m-torsion points

$$E[m] = \{P \in E = E(\overline{\mathbb{F}_p}) \mid [m]P = P_\infty\}$$

is a subgroup of E.

- We denote by

  $$E[m](L) = \{P \in E(L) \mid [m]P = P_\infty\}$$

  the group of L-rational m-torsion points.
- If $p \nmid m$ we have

  $$E[m] \cong \mathbb{Z}/m\mathbb{Z} \times Z/m\mathbb{Z}.$$

The embedding degree

Let $r \neq p$ be a large prime dividing $n = \#E(\mathbb{F}_p)$.
The embedding degree of E with respect to r is the smallest integer k s.t.

$$r \mid p^k - 1.$$

- This is equivalent to $r \mid \Phi_k(p)$, where Φ_k is the k-th cyclotomic polynomial. This follows from

$$X^k - 1 = \prod_{d \mid k} \Phi_d(X) = \Phi_k(X) \cdot \prod_{d \mid k, d \neq k} \Phi_d(X).$$

The embedding degree

- The embedding degree k is the order of p modulo r. Therefore
$$k \mid r - 1.$$
- For $k > 1$ the field $\mathbb{F}_{p^k}$ is the smallest extension of $\mathbb{F}_p$ which contains the group μ_r of r-th roots of unity,
- and for which $E(\mathbb{F}_{p^k})$ contains all r-torsion points, i.e.
$$E[r] \subseteq E(\mathbb{F}_{p^k}).$$

For crypto-sized curve E and prime divisor r the embedding degree is usually very large.

The Weil pairing

The Weil pairing is a map

$$\begin{aligned} e_r : E[r] \times E[r] &\rightarrow \mu_r \subseteq \mathbb{F}_{p^k}^*, \\ (P, Q) &\mapsto f_{r,P}(D_Q)/f_{r,Q}(D_P), \end{aligned}$$

- where $D_P \sim (P) - (P_\infty)$ and $D_Q \sim (Q) - (P_\infty)$ are divisors with disjoint support,
- $f_{r,P}$ and $f_{r,Q}$ are functions on the curve with divisors

$$\begin{aligned} (f_{r,P}) &= rD_P = r(P) - r(P_\infty), \\ (f_{r,Q}) &= rD_Q = r(Q) - r(P_\infty). \end{aligned}$$

The Weil pairing

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- For a divisor $D = \sum_{P \in E} n_P(P)$ and a function $f \in \overline{\mathbb{F}_p}(E)$, we can evaluate f at D by

$$f(D) = \prod_{P \in E} f(P)^{n_p}.$$

- The Weil pairing is bilinear, non-degenerate and alternating (i.e. $e_r(P, P) = 1$).

The MOV-FR attack

Theorem: Let $P \in E[r](\mathbb{F}_p)$. Then there exists a point $Q \in E[r]$ s.t. $e_r(P, Q)$ is a primitive r-th root of unity, i.e. a generator of μ_r.

- Let P, Q be the points from the theorem. Then the map
$$f : \langle P \rangle \longrightarrow \mu_r,\ R \mapsto e_r(R, Q)$$
is a group isomorphism.
- The map f 'reduces' the DLP on $E(\mathbb{F}_p)[r]$ to the DLP in $\mu_r \subseteq \mathbb{F}_{p^k}^*$: If $R = [m]P$ then
$$e_r(R, Q) = e_r([m]P, Q) = e_r(P, Q)^m.$$

The MOV-FR attack

$$
\begin{array}{rcl}
R & = & [m]P \\
 & \updownarrow & \\
e_r(R, Q) & = & e_r([m]P, Q) = e_r(P, Q)^m.
\end{array}
$$

- One can find m by solving the DLP in $\mathbb{F}_{p^k}^*$.
- This attack is only useful, if we can compute the Weil pairing efficiently,
- and if the DLP in $\mathbb{F}_{p^k}^*$ is easier than the DLP in $E(\mathbb{F}_p)$.

The Tate pairing

The Tate pairing is a map

$$\begin{aligned} \langle \cdot, \cdot \rangle_r : E[r](\mathbb{F}_{p^k}) \times E(\mathbb{F}_{p^k})/rE(\mathbb{F}_{p^k}) &\longrightarrow \mathbb{F}_{p^k}^*/(\mathbb{F}_{p^k}^*)^r, \\ (P, Q) &\longmapsto f_{r,P}(D_Q). \end{aligned}$$

- The divisor D_Q is equivalent to the divisor $(Q) - (P_\infty)$ and its support is disjoint from the support of $(f_{r,P}) = r(P) - r(P_\infty)$.
- The result must be interpreted as representing a class in $\mathbb{F}_{p^k}^*/(\mathbb{F}_{p^k}^*)^r$.
- Q is a representative of a class in $E(\mathbb{F}_{p^k})/rE(\mathbb{F}_{p^k})$.

The reduced Tate pairing

The reduced Tate pairing is a map

$$\begin{aligned} t_r : E[r](\mathbb{F}_p) \times E[r](\mathbb{F}_{p^k}) &\rightarrow \mu_r \subset \mathbb{F}_{p^k}^*, \\ (P, Q) &\mapsto f_{r,P}(Q)^{\frac{p^k-1}{r}}. \end{aligned}$$

- For the first group we restrict to $E[r](\mathbb{F}_p)$.
- If $r^2 \nmid n$ we may represent $E(\mathbb{F}_{p^k})/rE(\mathbb{F}_{p^k})$ by $E[r](\mathbb{F}_{p^k})$.
- For $k > 1$ we may replace D_Q by Q itself.
- Note that for $k > 1$ and $P \in E[r](\mathbb{F}_p)$ we have $t_r(P, P) = 1$.

The reduced Tate pairing

The reduced Tate pairing is a map

$$\begin{aligned} t_r : E[r](\mathbb{F}_p) \times E[r](\mathbb{F}_{p^k}) &\rightarrow \mu_r \subset \mathbb{F}_{p^k}^*, \\ (P, Q) &\mapsto f_{r,P}(Q)^{\frac{p^k-1}{r}}. \end{aligned}$$

- We obtain a unique pairing value in μ_r by raising $f_{r,P}(Q)$ to the power of $\frac{p^k-1}{r}$.
- This so called final exponentiation is an isomorphism $\mathbb{F}_{p^k}^*/(\mathbb{F}_{p^k}^*)^r \rightarrow \mu_r$.

Miller functions

To compute pairings we need to know the functions $f_{r,P}$ with divisor $r(P) - r(P_\infty)$.

- Let $f_{i,P},\ i \in \mathbb{Z}$ be a function on E which has a divisor

$$(f_{i,P}) = i(P) - ([i]P) - (i-1)(P_\infty).$$

  $f_{i,P}$ is called a Miller function.

- The special case $i = r$ leads to

$$(f_{r,P}) = r(P) - ([r]P) - (r-1)(P_\infty) = r(P) - r(P_\infty),$$

  since $[r]P = P_\infty$.

Miller's formula

Can we compute $f_{i+j,P}$ from $f_{i,P}$ and $f_{j,P}$?

- Compute the divisor of the product

$$
\begin{aligned}
(f_{i,P} f_{j,P}) &= i(P) - ([i]P) - (i-1)(P_\infty) \\
&\quad + j(P) - ([j]P) - (j-1)(P_\infty) \\
&= (i+j)(P) - ([i]P) - ([j]P) - (i+j-2)(P_\infty) \\
&= (i+j)(P) - ([i+j]P) - (i+j-1)(P_\infty) \\
&\quad + ([i+j]P) - ([i]P) - ([j]P) + (P_\infty) \\
&= (f_{i+j,P}) + ([i+j]P) - ([i]P) - ([j]P) + (P_\infty)
\end{aligned}
$$

- The sum of the divisors is 'almost' the divisor of $f_{i+j,P}$.

Miller's formula

Now have a look at the lines occuring in the addition $[i]P + [j]P = [i+j]P$.

- The first line l goes through $[i]P$, $[j]P$ and $-[i+j]P$, it has the divisor

$$(l) = ([i]P) + ([j]P) + (-[i+j]P) - 3(P_\infty).$$

- The second line v is a vertical line through $[i+j]P$ and $-[i+j]P$ with

$$(v) = ([i+j]P) + (-[i+j]P) - 2(P_\infty).$$

- Compute

$$(l) - (v) = ([i]P) + ([j]P) - ([i+j]P) - (P_\infty).$$

Miller's formula

- Remember

$$(f_{i,P}f_{j,P}) = (f_{i+j,P}) + ([i+j]P) - ([i]P) - ([j]P) + (P_\infty)$$

- and

$$(l) - (v) = ([i]P) + ([j]P) - ([i+j]P) - (P_\infty).$$

We get an equation of divisors

$$(f_{i+j,P}) = (f_{i,P}f_{j,P}) + (l) - (v).$$

- For the functions we get Miller's formula

$$f_{i+j,P} = f_{i,P}f_{j,P} \cdot l/v.$$

We can choose normalized functions, i.e. $f_{1,P} = 1$.

Computing pairings (Miller's algorithm)

We can use the special cases $i = j$ and $j = 1$ to compute the function $f_{r,P}$ in a square-&-multiply-like manner.

- Square step:

$$f_{2i,P} = f_{i,P}^2 \cdot l_{[i]P,[i]P}/v_{[2i]P}.$$

- Multiply step:

$$f_{i+1,P} = f_{i,P} f_{1,P} \cdot l_{[i]P,P}/v_{[i+1]P}.$$

- $l_{R,S}$: line through R and S, tangent if $R = S$,
  v_R: vertical line through R.

Computing pairings (Miller's algorithm)

Input: $P \in E[r](\mathbb{F}_p), Q \in E[r](\mathbb{F}_{p^k}), r = (r_m, \dots, r_0)_2$
Output: $f_{r,P}(Q)$

$R \leftarrow P$, $f \leftarrow 1$
for $(i \leftarrow m-1;\ i \geq 0;\ i--)$ **do**
  $f \leftarrow f^2 \frac{l_{R,R}(Q)}{v_{[2]R}(Q)}$
  $R \leftarrow [2]R$
  if $(r_i = 1)$ **then**
   $f \leftarrow f \frac{l_{R,P}(Q)}{v_{R+P}(Q)}$
   $R \leftarrow R + P$
  end if
end for
return f

Computing pairings (Miller's algorithm)

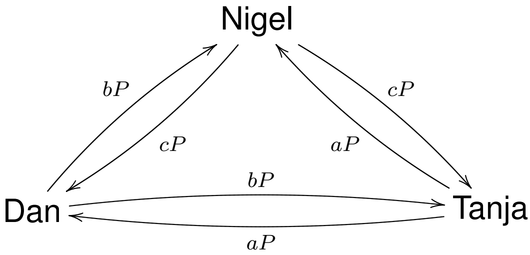
For Miller's algorithm we need arithmetic in $E(\mathbb{F}_p)$ and $\mathbb{F}_{p^k}$.

- If k is too large, we can't compute pairings this way.
- We need special curves with small k to be able to compute in $\mathbb{F}_{p^k}$.
- See tomorrow's talk for methods how to find such curves.

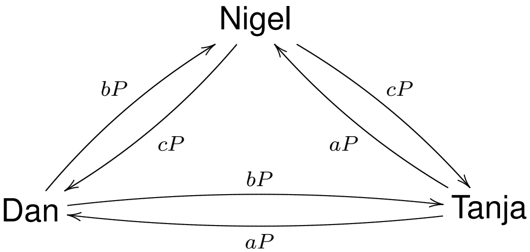
Tripartite key agreement

Tanja, Dan and Nigel would like to share a common secret key.

- They each choose a secret $a, b, c \in \mathbb{Z}_r$ resp.
- They compute aP, bP, cP resp. and send it to the other two.



Tripartite key agreement



- Using a pairing e the three can compute a common secret key using their secrets:

$$e(aP, bP)^c = e(bP, cP)^a = e(aP, cP)^b = e(P, P)^{abc}.$$

- Only one round of communication is needed.

Symmetric Pairings

If $k > 1$ we can use the reduced Tate pairing on supersingular curves to construct a symmetric pairing

$$e : E[r](\mathbb{F}_p) \times E[r](\mathbb{F}_p) \to \mu_r \subseteq \mathbb{F}_{p^k}^*,$$

s.t. $e(P, P) \neq 1$.

- Supersingular elliptic curves have $k \leq 6$.
- Supersingular elliptic curves have distortion maps.
- A distortion map is an endomorphism ϕ of E for which $\phi(P) \notin E(\mathbb{F}_p)$. If $E(\mathbb{F}_{p^k})$ has no points of order r^2 then

$$e(P, P) := t_r(P, \phi(P)) \neq 1.$$

BLS signatures

Using pairings it is possible to define a signature scheme with very short signatures.

- System parameters are the pairing

$$e : \langle P \rangle \times \langle Q \rangle \quad \longrightarrow \quad \mu_r \subseteq \mathbb{F}_{p^k}^*,$$

  points $P \in E[r](\mathbb{F}_p)$, $Q \in E[r](\mathbb{F}_{p^k})$ s.t. $e(P, Q) \neq 1$ and a hash function

$$H : \{0, 1\}^* \rightarrow E[r](\mathbb{F}_p).$$

BLS signatures

- To sign messages, Tanja chooses a private key $x_T \in \mathbb{Z}_r$ and publishes her public key $Q_T = [x_T]Q$.
- She signs the message $M \in \{0, 1\}^*$ by computing $H(M) \in E[r](\mathbb{F}_p)$ and the signature

$$\sigma = [x_T]H(M).$$

- To verify, anyone may take Q_T and check if

$$e(\sigma, Q) = e(H(M), Q_T).$$

- $e(\sigma, Q) = e([x_T]H(M), Q) = e(H(M), [x_T]Q) = e(H(M), Q_T)$.

BLS signatures

- The signature σ is just one point in $E[r](\mathbb{F}_p)$, so can be represented by 2 $\mathbb{F}_p$-elements.
- Compare this to the signatures from Tanja's 1st talk. There the signature was (R, S), where

$$R = [k]P,\ S = s_s m + kH([k]P) \mod r.$$

- This is 1 element of size r larger.
- If we represent points in $E(\mathbb{F}_p)$ by their x-coordinate only, this might be about half the size of the whole signature.

The Tate pairing is a bit slow...

Reducing the loop length - variants of the Tate pairing

It is possible to reduce the loop length in Miller's algorithm significantly and still get a pairing.

- Ate pairing:

$$\begin{aligned} \mathrm{ate} : E[r](\mathbb{F}_{p^k}) \times E[r](\mathbb{F}_p) &\longrightarrow \mu_r \subset \mathbb{F}_{p^k}^*, \\ (Q, P) &\mapsto f_{T,Q}(P)^{\frac{p^k-1}{r}}. \end{aligned}$$

  Here $T = t - 1$ where t is the trace of Frobenius, i.e. the number of bits in T is about half that of r.

Reducing the loop length - variants of the Tate pairing

- Twisted ate pairing: If E has a twist E' of degree d, we get a pairing

$$\begin{aligned} \mathrm{eta} : E[r](\mathbb{F}_p) \times E'[r](\mathbb{F}_{p^{k/d}}) &\rightarrow \mu_r \subset \mathbb{F}_{p^k}^*, \\ (P, Q') &\mapsto f_{T^e,P}(\phi(Q'))^{\frac{p^k-1}{r}}. \end{aligned}$$

  We have $T = t - 1$ and $T^e \equiv \zeta_m \mod r$, $e = k/m$, $m = \gcd(k, d)$. $\phi : E'[r](\mathbb{F}_{p^{k/d}}) \rightarrow E[r](\mathbb{F}_{p^k})$.

Reducing the loop length - variants of the Tate pairing

- There are other choices for the loop variable which even give shorter loops depending on the type of curves one is using.
- Shortest loops right now are of length $1/\varphi(k)$ times the length of r. Corresponding pairings are called optimal pairings.

For more information we refer to