Efficient Computation of Pairings on Elliptic Curves

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2 October 2009

Pairings

A pairing is a map

$$e : G_1 \times G_2 \to G_3$$

($(G_1, +), (G_2, +), (G_3, \cdot)$ finite abelian groups), which is

- *bilinear*,

$$\begin{aligned} e(P_1 + P_2, Q_1) &= e(P_1, Q_1)e(P_2, Q_1), \\ e(P_1, Q_1 + Q_2) &= e(P_1, Q_1)e(P_1, Q_2), \end{aligned}$$

- *non-degenerate*, given $0 \neq P \in G_1$ there is a $Q \in G_2$ with

$$e(P, Q) \neq 1,$$

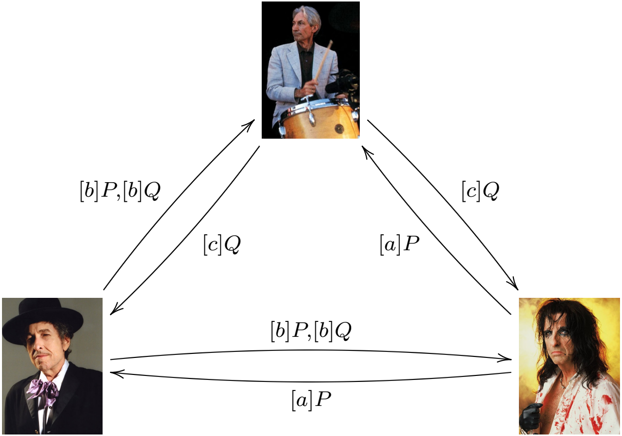
- *efficiently computable*.

Applications of pairings

- Attack DL-based cryptography on elliptic curves (Menezes-Okamoto-Vanstone-1993, Frey-Rück-1994) .
- Construct crypto systems with certain special properties:
  - One-round tripartite key agreement (Joux-2000),
  - Identity-based, non-interactive key agreement (Ohgishi-Kasahara-2000),
  - Identity-based encryption (Boneh-Franklin-2001),
  - Hierarchical IBE (Gentry-Silverberg-2002),
  - Short signatures (Boneh-Lynn-Shacham-2001).
  - Non-interactive proof systems (Groth-Sahai-2008)
  - much more ...

Tripartite key agreement (Joux-2000)

Alice, Bob, and Charlie choose secrets a, b, and c.



$$e([a]P,[b]Q)^c = e([b]P,[c]Q)^a = e([a]P,[c]Q)^b = e(P,Q)^{abc}$$

BLS signatures (Boneh-Lynn-Shacham-2001)

- System parameters:

$$e : G_1 \times G_2 \to G_3,$$

  elements $P \in G_1$, $Q \in G_2$ s.t. $e(P, Q) \neq 1$,
  and a hash function $H : \{0, 1\}^* \to G_1$.
- Alice's private key: $x_A \in \mathbb{Z}$, public key: $Q_A = [x_A]Q$.
- Signature of a message $M \in \{0, 1\}^*$: $\sigma = [x_A]H(M)$.
- Verification $e(\sigma, Q) = e(H(M), Q_A)$.
- Correctness: $e(\sigma, Q) = e([x_A]H(M), Q) = e(H(M), [x_A]Q) = e(H(M), Q_A)$.

Schedule of this talk

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(1) Elliptic Curves

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(2) Pairings on
(1) Elliptic Curves

Schedule of this talk

(3) Computation of
(2) Pairings on
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Schedule of this talk

(4) Efficient
(3) Computation of
(2) Pairings on
(1) Elliptic Curves

Elliptic Curves

Elliptic curves

Take an elliptic curve E over $\mathbb{F}_p$ ($p > 3$) with

- Weierstrass equation

$$E : y^2 = x^3 + ax + b,$$

- $E(\mathbb{F}_p) = \{(x, y) \in \mathbb{F}_p^2 : y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$,
- $n = \#E(\mathbb{F}_p) = p + 1 - t, \quad |t| \leq 2\sqrt{p}$,
- and $r \mid n$ a large prime divisor of n ($r \neq p$).
- For $\mathbb{F} \supseteq \mathbb{F}_p$:
  $E(\mathbb{F}) = \{(x, y) \in \mathbb{F}^2 : y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$,
- $E = E(\overline{\mathbb{F}_p})$, $\overline{\mathbb{F}_p}$ an algebraic closure of $\mathbb{F}_p$.
- E is an abelian group (written additively).

Torsion points and embedding degree

The set of r-torsion points on E is

$$E[r] = \{P \in E \mid [r]P = \mathcal{O}\}.$$

Since $r \mid \#E(\mathbb{F}_p)$, we have $E(\mathbb{F}_p)[r] \neq \emptyset$.
The embedding degree of E w.r.t. r is the smallest integer k with

$$r \mid p^k - 1.$$

For $k > 1$ we have

$$E[r] \subset E(\mathbb{F}_{p^k}),$$

i. e. $E(\mathbb{F}_p)[r] \subseteq E(\mathbb{F}_{p^k})[r] = E[r]$.

Pairings on Elliptic Curves

The reduced Tate pairing

The reduced Tate pairing

$$\begin{aligned} t_r : E(\mathbb{F}_{p^k})[r] \times E(\mathbb{F}_{p^k})/[r]E(\mathbb{F}_{p^k}) &\rightarrow \mu_r \subset \mathbb{F}_{p^k}^*, \\ (P, Q) &\mapsto f_{r,P}(Q)^{\frac{p^k-1}{r}}. \end{aligned}$$

defines a non-degenerate, bilinear map, where

- μ_r is the group of r-th roots of unity in $\mathbb{F}_{p^k}^*$,
- $f_{r,P}$ is a function with divisor $(f_{r,P}) = r(P) - r(\mathcal{O})$.

For $P \in E(\mathbb{F}_p)[r]$, we have $t_r(P, P) = 1$, take $Q \notin \langle P \rangle$.

Three groups

Assume $r^2 \nmid \#E(\mathbb{F}_p)$, $k > 1$. Define the following groups:

- $G_1 = E(\mathbb{F}_{p^k})[r] \cap \ker(\phi_p - [1]) = E(\mathbb{F}_p)[r]$,
- $G_2 = E(\mathbb{F}_{p^k})[r] \cap \ker(\phi_p - [p])$,
- $G_3 = \mu_r \subset \mathbb{F}_{p^k}^*$.

ϕ_p is the p-power Frobenius on E, i. e. $\phi_p(x, y) = (x^p, y^p)$.
Let

$$G_1 = \langle P \rangle, \quad G_2 = \langle Q \rangle.$$

We have $E(\mathbb{F}_{p^k})[r] = G_1 \oplus G_2$, and we compute the Tate pairing as

$$\begin{aligned} t_r : G_1 \times G_2 &\rightarrow G_3, \\ (P, Q) &\mapsto f_{r,P}(Q)^{\frac{p^k - 1}{r}}. \end{aligned}$$

G_1, G_2, and G_3 are cyclic groups of prime order r.

Computation of
Pairings on Elliptic Curves

Computing the pairing

There are two parts:

1. compute $f_{r,P}(Q)$,
2. the final exponentiation to the power $(p^k - 1)/r$.

For the first part, consider Miller functions $f_{i,P}$, $i \in \mathbb{Z}$.
These are functions with divisor

- $(f_{i,P}) = i(P) - ([i]P) - (i-1)(\mathcal{O})$.

Then

- $(f_{r,P}) = r(P) - ([r]P) - (r-1)(\mathcal{O}) = r(P) - r(\mathcal{O})$.

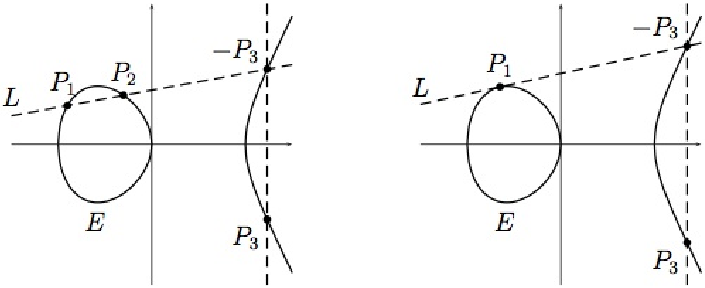
Miller functions and line functions

Miller functions can be computed recursively with

- $f_{1,P} = 1$,
- $f_{2i,P} = f_{i,P}^2 \cdot l_{[i]P,[i]P}/v_{[2i]P}$,
- $f_{i+1,P} = f_{i,P} \cdot l_{[i]P,P}/v_{[i+1]P}$,

where

- l_{P_1,P_2}: line through P_1 and P_2, tangent if $P_1 = P_2$,
  v_{P_1}: vertical line through P_1.



Miller's algorithm

Input: $P \in G_1, Q \in G_2, r = (r_m, \ldots, r_0)_2$

Output: $t_r(P, Q) = f_{r,P}(Q)^{\frac{p^k-1}{r}}$

$R \leftarrow P$, $f \leftarrow 1$

for $(i \leftarrow m-1;\ i \geq 0;\ i--)$ **do**

$f \leftarrow f^2 \frac{l_{R,R}(Q)}{v_{[2]R}(Q)}$

$R \leftarrow [2]R$

if $(r_i = 1)$ **then**

$f \leftarrow f \frac{l_{R,P}(Q)}{v_{R+P}(Q)}$

$R \leftarrow R + P$

end if

end for

$f \leftarrow f^{\frac{p^k-1}{r}}$

return f

Specific parameters – pairing-friendly curves

- The embedding degree k needs to be small ($1 < k \leq 50$), to be able to do computations at all.
- DLPs must be hard in all three groups.
- For efficiency reasons balance the security as much as possible.
- Define $\rho = \log(p)/\log(r)$.

Security level (bits)	Extension field size of p^k (bits)	EC base point order r (bits)	ratio $\rho \cdot k$
80	1024	160	6.40
112	2048	224	9.14
128	3072	256	12.00
192	7680	384	20.00
256	15360	512	30.00

NIST recommendations

My favorite examples... BN curves (Barreto-N., 2005)

BN curves can be found easily and are ideal for the 128-bit security level.
If $u \in \mathbb{Z}$ such that

$$
\begin{aligned}
p = p(u) &= 36u^4 + 36u^3 + 24u^2 + 6u + 1, \\
n = n(u) &= 36u^4 + 36u^3 + 18u^2 + 6u + 1
\end{aligned}
$$

are both prime, then there exists an elliptic curve

- with equation $E : y^2 = x^3 + b,\ b \in \mathbb{F}_p$,
- $r = n = \#E(\mathbb{F}_p)$ is prime, i. e. $\rho \approx 1$,
- the embedding degree is $k = 12$.
- BNtiny: $u = -1, p = 19, n = 13, E : y^2 = x^3 + 3$. $P = (1, 2) \in E(\mathbb{F}_p)$.

Efficient Computation of Pairings on Elliptic Curves

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Final exponentiation (easy part)

- Choose k even, then the final exponent is

$$\frac{p^k - 1}{r} = (p^{k/2} - 1)\frac{p^{k/2} + 1}{r}.$$

Note that $r \nmid p^{k/2} - 1$, therefore $r \mid p^{k/2} + 1$.

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- Then $f = f_0 + f_1\alpha$ with $f_0, f_1 \in \mathbb{F}_{p^{k/2}}$, computing $(f_0 + f_1\alpha)^{p^{k/2}} = f_0 - f_1\alpha$ is almost for free,
- and $(f_0 + f_1\alpha)^{p^{k/2}-1} = (f_0 - f_1\alpha)/(f_0 + f_1\alpha)$.

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Denominator elimination

- Since k is even, all points $Q \in G_2$ have a special form, in particular the x-coordinate $x_Q \in \mathbb{F}_{p^{k/2}}$.
- The value of the vertical line function $v_R(Q) = x_Q - x_R \in \mathbb{F}_{p^{k/2}}$.
- The first part of the final exponentiation thus gives

$$v_R(Q)^{p^{k/2}-1} = 1.$$

- Remove all denominators in Miller's algorithm.
- Similarly, all values in proper subfields of $\mathbb{F}_{p^k}$ are mapped to 1 by the final exponentiation.

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Doubling and addition steps

$$\begin{aligned} \mathrm{DBL}: &\quad f \leftarrow f^2 \cdot l_{R,R}(Q), \quad R \leftarrow [2]R \\ \mathrm{ADD}: &\quad f \leftarrow f \cdot l_{R,P}(Q), \quad R \leftarrow R + P \end{aligned}$$

These steps include multiplications/squarings in $\mathbb{F}_{p^k}$, computations in $\mathbb{F}_p$ for the line coefficients, and curve arithmetic in $E(\mathbb{F}_p)$.

- Line functions correspond to the lines in the point doubling/addition,
- reuse intermediate results of point additions for line function coefficients,
- use projective coordinates to avoid inversions.

What about Edwards curves?

Edwards curves provide extremely fast curve arithmetic.
Can we use this advantage for pairings?

$$E_d : x^2 + y^2 = 1 + dx^2y^2$$

- Edwards group law

$$(x_1, y_1) + (x_2, y_2) = (x_3, y_3),$$

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2} \text{ and } y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

- Neutral element is $\mathcal{O} = (0, 1)$, $-(x_1, y_1) = (-x_1, y_1)$.
  $\mathcal{O}' = (0, -1)$ has order 2; $(1, 0), (-1, 0)$ have order 4.
- Two points at infinity $\Omega_1 = (1 : 0 : 0)$, $\Omega_2 = (0 : 1 : 0)$ with multiplicity 2.

Pairings on Edwards curves

- Line functions do not work: Edwards equation has degree 4, so expect 4 intersection points.
- Quadratic functions: 8 intersection points.
- Replace line by the conic C passing through the 5 points $P_1, P_2, \mathcal{O}', \Omega_1,$ and Ω_2.
  Only *one more* intersection point.

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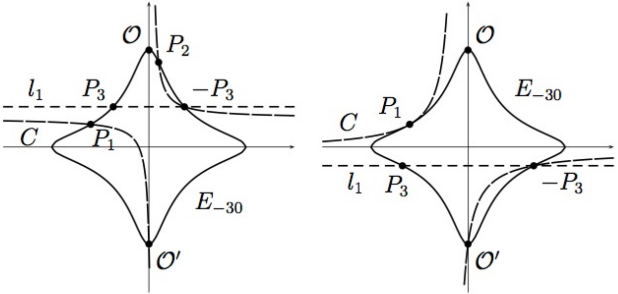

Pairings on Edwards curves

- Can do Miller's algorithm as before,
- only replace line functions by quadratic functions described by the above conic.
- Comparison of costs for computing the coefficients of lines or conics and the double or sum of points:

	DBL	mADD	ADD
Jacobian coord.	$1\mathbf{m} + 11\mathbf{s} + 1\mathbf{m_a}$	$6\mathbf{m} + 6\mathbf{s}$	$15\mathbf{m} + 6\mathbf{s}$
Jacobian ($a = -3$)	$6\mathbf{m} + 5\mathbf{s}$	$6\mathbf{m} + 6\mathbf{s}$	$15\mathbf{m} + 6\mathbf{s}$
Jacobian ($a = 0$, e.g. BN curves)	$3\mathbf{m} + 8\mathbf{s}$	$6\mathbf{m} + 6\mathbf{s}$	$15\mathbf{m} + 6\mathbf{s}$
Edwards	$6\mathbf{m} + 5\mathbf{s}$	$12\mathbf{m}$	$14\mathbf{m}$

Miller's algorithm

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$R \leftarrow R + P$

end if

end for

$f \leftarrow f^{p^{k/2}-1} = f^{p^{k/2}}/f$

$f \leftarrow f^{\frac{p^{k/2}+1}{r}}$

return f

The Miller loop

- If possible, choose r with low hamming weight.
- If not, maybe use Non-Adjacent-Form (NAF):
  $r = (r_{m+1}, \ldots, r_0)_{\mathrm{NAF}},\ r_i \in \{-1, 0, 1\}$

for $(i \leftarrow m;\ i \geq 0;\ i--)$ **do**
  $f \leftarrow f^2 \cdot l_{R,R}(Q)$
  $R \leftarrow [2]R$
  if $(r_i = 1)$ **then**
    $f \leftarrow f \cdot l_{R,P}(Q)$
    $R \leftarrow R + P$
  end if
  if $(r_i = -1)$ **then**
    $f \leftarrow f \cdot l_{R,-P}(Q)$
    $R \leftarrow R - P$
  end if
end for

Loop shortening - eta pairing

Suppose E has a twist of degree δ and $\delta \mid k$. Let $e = k/\delta$ and $T_e = (t-1)^e \mod r$.

- It turns out that the map

$$\begin{aligned} \eta_{T_e} : G_1 \times G_2 &\longrightarrow G_3, \\ (P, Q) &\longmapsto f_{T_e,P}(Q)^{(p^k-1)/r}. \end{aligned}$$

  is a pairing, called the eta pairing.
- One can take $T_e^j \mod r$ for $1 \le j \le \delta - 1$ instead of T_e. Choose the shortest non-trivial power.

Loop shortening - ate pairing

Let $T = t - 1$.

- The map

$$\begin{aligned} a_T : G_2 \times G_1 &\to G_3, \\ (Q, P) &\mapsto f_{T,Q}(P)^{(p^k-1)/r}. \end{aligned}$$

  is a pairing, called the ate pairing.
- As for the eta pairing, we can replace T by $T^j \mod r$ for $1 \leq j \leq k-1$ to possibly get a shorter loop.
- Note that groups are swapped. Curve arithmetic in Miller's algorithm must now be done over a field extension.

Miller's algorithm

Input: $P \in G_1, Q \in G_2, r = (r_m, \ldots, r_0)_2$

Output: $t_r(P, Q) = f_{r,P}(Q)^{\frac{p^k - 1}{r}}$

$R \leftarrow P$, $f \leftarrow 1$

for $(i \leftarrow m - 1;\ i \geq 0;\ i - -)$ **do**

$f \leftarrow f^2 \cdot l_{R,R}(Q)$

$R \leftarrow [2]R$

if $(r_i = 1)$ **then**

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$f \leftarrow f^{p^{k/2} - 1} = f^{p^{k/2}} / f$

$f \leftarrow f^{\frac{p^{k/2} + 1}{r}}$

return f

Final exponentiation (hard part)

Let Φ_k be the kth cyclotomic polynomial.

- The embedding degree condition

$$r \mid p^k - 1,\ r \nmid p^m - 1 \text{ for } m < k$$

  is equivalent to $r \mid \Phi_k(p)$.
- $\Phi_k(p) \mid p^{k/2} + 1$.
- The second part of the final exponent can be written as

$$\frac{p^{k/2} + 1}{r} = \frac{p^{k/2} + 1}{\Phi_k(p)} \cdot \frac{\Phi_k(p)}{r}.$$

Final exponentiation (hard part)

k	$\Phi_k(p)$	$(p^{k/2}+1)/\Phi_k(p)$
6	$p^2 - p + 1$	$p + 1$
10	$p^4 - p^3 + p^2 - p + 1$	$p + 1$
12	$p^4 - p^2 + 1$	$p^2 + 1$
16	$p^8 + 1$	1
18	$p^6 - p^3 + 1$	$p^3 + 1$
24	$p^8 - p^4 + 1$	$p^4 + 1$
30	$p^8 + p^7 - p^5 - p^4$	$p^7 - p^6 + p^5$
	$-p^3 + p + 1$	$+p^2 - p + 1$

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30	$p^8 + p^7 - p^5 - p^4$	$p^7 - p^6 + p^5$
	$-p^3 + p + 1$	$+p^2 - p + 1$

- Example $k = 12$:

$$\frac{p^6+1}{r} = (p^2+1) \cdot \frac{p^4-p^2+1}{r}.$$

- Compute $f^{(p^6+1)/r} = ((f^p)^p \cdot f)^{(p^4-p^2+1)/r}$.

p-power Frobenius

Example BN curves with $k = 12$:
note $p \equiv 1 \pmod 6$.

- $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$, $\alpha^2 = \beta$
  Then an element $f \in \mathbb{F}_{p^2}$ can be written as $f = f_0 + f_1\alpha$ with $f_0, f_1 \in \mathbb{F}_p$, thus

$$f^p = (f_0 + f_1\alpha)^p = f_0 - f_1\alpha.$$

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- $\mathbb{F}_{p^6} = \mathbb{F}_{p^2}(w)$, $w^3 = \xi$ for $\xi \in \mathbb{F}_{p^2}$ not a cube, not a square
  Write $f = f_0 + f_1 w + f_2 w^2$ with $f_0, f_1, f_2 \in \mathbb{F}_{p^2}$. Then

$$f^p = f_0^p + f_1^p w_p w + f_2^p w_p^2 w^2,$$

  where $w_p = w^{p-1} = \xi^{\frac{p-1}{3}} \in \mathbb{F}_{p^2}$.

p-power Frobenius

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- One p-power Frobenius $f \mapsto f^p$ for an element in $\mathbb{F}_{p^{12}}$ can be done with 7 multiplications in $\mathbb{F}_{p^2}$.
- A plain square-and-multiply exponentiation needs at least $\log(p)$ squarings in $\mathbb{F}_{p^{12}}$.

The new hard part

It remains to compute a power to the exponent $\frac{\Phi_k(p)}{r}$.
For BN curves:

$$\frac{\Phi_k(p)}{n} = \frac{p^4 - p^2 + 1}{n} = p^3 + l_2 p^2 + l_1 p + l_0,$$

with

$$\begin{aligned} l_2 &= 6u^2 + 1, \\ l_1 &= -36u^3 - 18u^2 - 12u + 1, \\ l_0 &= -36u^3 - 30u^2 - 18u + 2. \end{aligned}$$

Multi-exponentiation

To compute $f^{(p^4-p^2+1)/n}$,

- first obtain f^p, f^{p^2}, f^{p^3} by three Frobenius applications,
- then compute

$$f^{l_0+l_1p+l_2p^2} = f^{l_0}(f^p)^{l_1}(f^{p^2})^{l_2}$$

  with a multi-exponentiation,
- and finally

$$f^{l_0+l_1p+l_2p^2+p^3} = f^{l_0}(f^p)^{l_1}(f^{p^2})^{l_2}f^{p^3}.$$

The final slide... cheap pairings...

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